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Korovkin Approximation for Weighted Set-Valued Functions

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We investigate an approximation problem in weighted spaces of continuous functions whose domain is a locally compact Hausdorff space and whose values are non-empty compact convex subsets of a locally convex space. We prove a Korovkin type approximation theorem for monotone linear operators on such spaces and generalize results from earlier work which deals with compact domains and compact convex subsets of a Frechét space, resp. of $\mathbb{R}^n$. © 1999 Academic Press

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1. WEIGHTED SPACES OF REAL-VALUED FUNCTIONS

Throughout this paper, let X be a locally compact Hausdorff space and $C(X)$ the space of all continuous real-valued functions on X. A function $f: X \to \mathbb{R}$ is said to *vanish at infinity* if for every $\varepsilon > 0$ the set $\{x \in X \mid |f(x)| \geqslant \varepsilon\}$ is relatively compact in X. According to Nachbin [8] and Prolla [9] a family $\mathscr{W}$ of non-negative upper semicontinuous functions on X is called a *family of weights* if for all $w_1, w_2 \in \mathscr{W}$ there are $w_3 \in \mathscr{W}$ and $\rho > 0$ such that $w_1 \leqslant \rho w_3$ and $w_2 \leqslant \rho w_3$. With any family of weights $\mathscr{W}$ we associate the subspace of $C(X)$

$$C_{\mathscr{W}}(X) = \{f \in C(X) \mid wf \text{ vanishes at infinity for all } w \in \mathscr{W}\}.$$

Together with the locally convex topology generated by the seminorms

$$p_w(f) = \sup\{|wf(x)| \mid x \in X\}$$

for $w \in \mathscr{W}$ and $f \in C_{\mathscr{W}}(X)$ we call $C_{\mathscr{W}}(X)$ a *weighted space of functions*. We shall mention a few examples and refer to [8–10] for details.

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1.1. EXAMPLES. (a) If $\mathscr{W}$ consists of the constant function $w \equiv 1$, then $C_{\mathscr{W}}(X) = C_0(X)$, the space of all functions in $C(X)$ that vanish at infinity, endowed with the supremum norm.

(b) If $\mathscr{W} = C_0^+(X) = \{w \in C_0(X) \mid w \geqslant 0\}$, then $C_{\mathscr{W}}(X) = C_B(X)$, the space of all bounded functions in $C(X)$. The weighted topology (called the *strict* topology) is generally coarser than the supremum norm topology on $C_B(X)$.

(c) If $\mathscr{W}$ consists of the characteristic functions of all compact subsets of X, then $C_{\mathscr{W}}(X) = C(X)$ with the topology of compact convergence.

(d) If $\mathscr{W}$ consists of the characteristic functions of all finite subsets of X, then $C_{\mathscr{W}}(X) = C(X)$ with the topology of pointwise convergence.

(e) If $\mathscr{W} = C^+(X) = \{w \in C(X) \mid w \geqslant 0\}$, then $C_{\mathscr{W}}(X) = C_c(X)$, the space of all functions with compact support in $C(X)$. The weighted topology is generally finer than the supremum norm but coarser than the inductive limit topology on $C_c(X)$.

The weighted spaces $C_{\mathscr{W}}(X)$ are endowed with the pointwise order for functions. Korovkin type theorems deal with approximation processes modeled by equicontinuous nets $(T_\alpha)_{\alpha \in \mathscr{A}}$ of positive linear operators $T_\alpha \colon C_{\mathscr{W}}(X) \to C_{\mathscr{W}}(X)$. For a subset $\mathscr{M}$ of $C_{\mathscr{W}}(X)$, the *Korovkin closure* $\mathscr{K}(\mathscr{M})$ of $\mathscr{M}$ consists of all functions $f \in C_{\mathscr{W}}(X)$ such that $T_\alpha(f)$ converges to f in the weighted topology of $C_{\mathscr{W}}(X)$ whenever $(T_\alpha)_{\alpha \in \mathscr{A}}$ is an equicontinuous net of positive linear operators on $C_{\mathscr{W}}(X)$ and $T_\alpha(m)$ converges to m for all $m \in \mathscr{M}$.

We shall cite Theorem 2.1 from [10]. For the special case of $C_0(X)$ with the supremum norm, i.e., the case of our Example 1.1(a), it is due to Bauer and Donner [2]. By span($\mathscr{M}$) we denote the linear span of $\mathscr{M}$, by $M_B(X)$ the space of all finite regular Borel measures on X, and by $M_B^+(X)$ the positive cone in $M_B(X)$. We set

$$X_0 = \{x \in X \mid w(x) > 0 \text{ for some } w \in \mathscr{W}\}.$$

1.2. THEOREM. *Let X be a locally compact Hausdorff space, and let $\mathscr{W}$ be a family of weight functions on X. Let $\mathscr{M}$ be a subset of $C_{\mathscr{W}}(X)$. For a function $f \in C_{\mathscr{W}}(X)$ the following are equivalent*:

(a) $f \in \mathscr{K}(\mathscr{M})$.

(b) *For every* $x \in X_0$

$$f(x) = \sup_{\substack{w \in \mathscr{W} \\ \varepsilon > 0}} \inf \{m(x) \mid m \in \operatorname{span}(\mathscr{M}),\ wf \leqslant wm + \varepsilon\}$$

$$= \inf_{\substack{w \in \mathscr{W} \\ \varepsilon > 0}} \sup \{m(x) \mid m \in \operatorname{span}(\mathscr{M}),\ wm \leqslant wf + \varepsilon\}.$$

(c) *For every* $x \in X_0$, $\phi \in M_B^+(X)$ *and* $w \in \mathcal{W}$

$$\int_X wm\, d\phi = m(x) \qquad \textit{for all} \quad m \in \mathcal{M} \quad \textit{implies} \quad \int_X wf\, d\phi = f(x).$$

A brief inspection of the proof in [10] reveals that for the validity of the implications (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (a) the operators T_α need to be defined only on a subcone $D_{\mathcal{W}}(X)$ of $C_{\mathcal{W}}(X)$ that contains both $\mathcal{M}$ and the function f. The requirements for operators on a cone, however, need to be adapted as follows: Linearity means that an operator $T: D_{\mathcal{W}}(X) \to C_{\mathcal{W}}(X)$ is additive and positively homogeneous. Continuity and monotony are combined in the following condition:

(C) *For every* $w \in \mathcal{W}$ *there are* $w' \in \mathcal{W}$ *and* $\rho > 0$ *such that for all* $f, g \in D_{\mathcal{W}}(X)$

$$wT_\alpha(f) \leqslant wT_\alpha(g) + 1 \qquad \textit{whenever} \quad w'f \leqslant w'g + \rho.$$

Equicontinuity for a net $(T_\alpha)_{\alpha \in A}$ of such operators means that the weight function w' and the constant $\rho > 0$ may be chosen simultaneously for all operators T_α. Any linear operator satisfying (C) may be extended to the linear span of $D_{\mathcal{W}}(X)$ under preservation of (C). We refer to [7] for a systematical introduction of continuous linear operators on (locally convex) cones.

A *Korovkin system* for $D_{\mathcal{W}}(X)$ is a subset $\mathcal{M}$ of $D_{\mathcal{W}}(X)$ such that $D_{\mathcal{W}}(X) \subset \mathcal{K}(\mathcal{M})$. Condition (c) of Theorem 1.2 yields the following criterion which is a reformulation of Corollary 2.3 in [10]:

1.3. Proposition. *Let* $\mathcal{M}$ *be a subset of* $C_{\mathcal{W}}(X)$. *Suppose that* $\mathcal{M}$ *contains a function that does not vanish on all of* X_0 *and that for every pair of distinct points* $x, y \in X_0$ *there is* $m_{x,y} \in \operatorname{span}(\mathcal{M})$ *such that*

$$m_{x,y} \geqslant 0 \quad \textit{on} \quad X_0, \qquad m_{x,y}(x) = 0, \qquad \textit{and} \qquad m_{x,y}(y) > 0.$$

Then $\mathcal{M}$ *is a Korovkin system for* $C_{\mathcal{W}}(X)$.

We shall say that a subset $\mathcal{E}$ of non-negative functions in $C_{\mathcal{W}}(X)$ is a *unit family* if for every $x \in X_0$ there is $e_x \in \mathcal{E}$ such that $e_x(x) > 0$. A subset $\mathcal{S} \subset C_{\mathcal{W}}(X)$ *separates the points of* X_0 if for all $x \neq y \in X_0$ there is $s \in \mathcal{S}$ such that $s(x) \neq s(y)$. For any pair of such families $\mathcal{E}$ and $\mathcal{S}$ the functions

$$\mathcal{M} = \{e, es, es^2 \mid e \in \mathcal{E}, s \in \mathcal{S}\}$$

fulfill the above criterion, hence form a Korovkin system for $C_{\mathscr{W}}(X)$: $\mathscr{E}$ contains a non-zero function, and the condition in Proposition 1.3 is satisfied by functions

$$y \mapsto e(y)(s(y) - s(x))^2.$$

For $X = [0, 1]$, $\mathscr{W} = \{1\}$, $\mathscr{E} = \{1\}$, and $\mathscr{S} = \{x\}$ this leads to the classical Korovkin system $\mathscr{M} = \{1, x, x^2\}$ for $C([0, 1])$. Many more examples may be found in [1, 2, 10]. Furthermore, condition (b) of Theorem 1.2 yields a Stone–Weierstrass type result (Corollary 2.2 in [10]): The vector sublattice generated by a Korovkin system $\mathscr{M}$ for $C_{\mathscr{W}}(X)$ is seen to be dense in $C_{\mathscr{W}}(X)$ with respect to its weighted topology.

2. REPRESENTATIONS OF COMPACT CONVEX SETS

In the following, let E be a real locally convex topological vector space, E' its dual endowed with the weak* topology. For a subset A of E, its *polar* in E' is $A^\circ = \{\mu \in E' \mid \mu(a) \leqslant 1 \text{ for all } a \in A\}$. The polar in E of a subset of E' is correspondingly defined. We consider the usual addition and multiplication of sets.

Let $\mathfrak{V}$ be a family (not necessarily a base) of closed convex neighborhoods of the origin in E, directed downward for set inclusion. A subset A of E is said to be *precompact relative to* $\mathfrak{V}$ if for all $V \in \mathfrak{V}$ and $\varepsilon > 0$ there are $a_1, ..., a_n \in A$ such that

$$A \subset \bigcup_{i=1}^{n} (a_i + \varepsilon V).$$

We shall denote the cone of all non-empty convex subsets of E that are precompact relative to $\mathfrak{V}$ by $\mathscr{C}_{\mathfrak{V}}(E)$ and by $\mathscr{D}_{\mathfrak{V}}(E)$ any subcone thereof. Every $A \in \mathscr{C}_{\mathfrak{V}}(E)$ is bounded relative to $\mathfrak{V}$, that is for every $V \in \mathfrak{V}$ there is $\rho > 0$ such that $A \subset \rho V$.

We proceed to represent the elements of $\mathscr{C}_{\mathfrak{V}}(E)$ as continuous real-valued functions on a suitable locally compact Hausdorff space Y. For a fixed $V \in \mathfrak{V}$ let

$$Y_V = \{(\mu, V) \mid 0 \neq \mu \in V^\circ\},$$

endowed with the weak topology of the injection

$$(\mu, V) \mapsto \mu : Y_V \to E'.$$

Thus Y_V is homeomorphic to $V^\circ \backslash \{0\}$ and therefore locally compact and Hausdorff. Now let

$$Y = \bigoplus_{V \in \mathfrak{B}} Y_V$$

be the topological sum of the disjoint spaces Y_V. A subset of Y is open (closed) if any only if its intersection with every Y_V is open (closed) in Y_V. As a sum of locally compact Hausdorff spaces Y is itself locally compact and Hausdorff.

Now with every set $A \in \mathscr{C}_{\mathfrak{B}}(E)$ we associate a real-valued function f_A on Y defined by

$$f_A(\mu, V) = \sup \{\mu(a) \mid a \in A\}$$

for all $(\mu, V) \in Y$. This function is seen to be continuous on Y, as for any fixed $V \in \mathfrak{B}$ and $\varepsilon > 0$ there are $a_1, ..., a_n \in A$ such that $A \subset \bigcup_{i=1}^{n} (a_i + \varepsilon U)$. The function $f'(\mu, V) = \max_{i=1}^{n} \mu(a_i)$ is obviously continuous on Y_V and

$$f'(\mu, V) \leqslant f_A(\mu, V) \leqslant f'(\mu, V) + \varepsilon$$

holds whenever $\mu \in V^\circ$. This shows that f_A itself is continuous on Y_V. But continuity an all of the spaces Y_V implies continuity on their topological sum. Next we observe that the mapping

$$A \mapsto f_A : \mathscr{C}_{\mathfrak{B}}(E) \to C(Y)$$

is additive and positively homogeneous, i.e. $f_{A+B} = f_A + f_B$ and $f_{\alpha A} = \alpha f_A$ for all $A, B \in \mathscr{C}_{\mathfrak{B}}(E)$ and $\alpha \geqslant 0$. If C is the convex hull of the sets $A, B \in \mathscr{C}_{\mathfrak{B}}(E)$, then f_C is the (pointwise) maximum of the functions f_A and f_B. Furthermore, if $V \in \mathfrak{B}$ and v denotes the characteristic function on Y of the subset Y_V, then for $A, B \in \mathscr{C}_{\mathfrak{B}}(E)$ and $\rho \geqslant 0$

$$v f_A \leqslant v f_B + \rho \quad \text{if and only if} \quad A \subset B + \rho' V \quad \text{for all } \rho' > \rho.$$

To verify this claim, let $A \subset B + \rho' V$ for any $\rho' > \rho$. Then for all $a \in A$ there are $b \in B$ and $v \in V$ such that $a = b + \rho' v$, hence $\mu(a) \leqslant \mu(b) + \rho'$ for all $\mu \in V^\circ$. This shows $f_A(\mu, V) \leqslant f_B(\mu, V) + \rho'$ for all $(\mu, V) \in Y_V$. If on the other hand $A \not\subset B + \rho' V$ for some $\rho' > \rho$, then $a \notin B + \rho' V$ for some $a \in A$, that is $(a - B) \cap \rho' V = \varnothing$. By the Hahn–Banach separation theorem there is $\mu \in E'$ such that $\mu(\rho' v) \leqslant \rho' \leqslant \mu(a) - \mu(b)$ for all $v \in V$ and $b \in B$. This shows $0 \neq \mu \in V^\circ$. Then $(\mu, V) \in Y_V$ and $\rho' \leqslant f_A(\mu, V) - f_B(\mu, V)$. Thus $f_A(\mu, V) > f_B(\mu, V) + \rho$, indeed.

The embedding $A \mapsto f_A$ is therefore monotone if we endow $\mathscr{C}_{\mathfrak{V}}(E)$ with the set inclusion as order, and one-to-one on every subcone $\mathscr{D}_{\mathfrak{V}}(E)$ of $\mathscr{C}_{\mathfrak{V}}(E)$ whose elements are closed sets in the topology on E that is created by the neighborhoods in $\mathfrak{V}$.

For every $V \in \mathfrak{V}$ the set Y_V is both open and closed in Y. Its characteristic function v is therefore continuous on Y. Moreover, for every $A \in \mathscr{C}_{\mathfrak{V}}(E)$ the function vf_A vanishes at infinity. Indeed, if $f_A(\mu, V) \geqslant \varepsilon$ for some $\varepsilon > 0$, then $(\mu, V) \in Y_V$ and the continuity of the function f_A on Y_V shows that the set $\{(\mu, V) \in Y_c \mid f_A(\mu, V) \geqslant \varepsilon\}$ is compact.

We may now introduce suitable weighted topologies on $\mathscr{C}_{\mathfrak{V}}(E)$: The set $\mathscr{V}$ of characteristic functions v corresponding to the neighborhoods $V \in \mathfrak{V}$ is directed upward and forms a family of weights on Y. Via the embedding $A \mapsto f_A$ the cone $\mathscr{C}_{\mathfrak{V}}(E)$ may be considered as a subcone of $C_{\mathscr{V}}(Y)$. The *weighted topology* on $\mathscr{C}_{\mathfrak{V}}(E)$ corresponding to the weighted topology of $C_{\mathscr{V}}(Y)$ is induced by the semimetrics

$$d_V(A, B) = \inf\{\rho \geqslant 0 \mid A \subset B + \rho V \text{ and } B \subset A + \rho V\}$$

for $A, B \in \mathscr{C}_{\mathfrak{V}}(E)$ and $V \in \mathfrak{V}$. We call $\mathfrak{V}$ a *family of weights on* $\mathscr{C}_{\mathfrak{V}}(E)$ and for any subcone $\mathscr{D}_{\mathfrak{V}}(E)$ of $\mathscr{C}_{\mathfrak{V}}(E)$ we call $(\mathscr{D}_{\mathfrak{V}}(E), \mathfrak{V})$ a *weighted cone of sets*. Let us illustrate this with a few examples:

2.1. EXAMPLES. (a) If E is a normed space with unit ball $\mathbb{B}$ and $\mathfrak{V} = \{\mathbb{B}\}$, then $\mathscr{C}_{\mathfrak{V}}(E)$ carries the topology of the *Hausdorff metric*

$$d_{\mathbb{B}}(A, B) = \inf\{\rho \geqslant 0 \mid A \subset B + \rho\mathbb{B} \text{ and } B \subset A + \rho\mathbb{B}\}$$

for $A, B \in \mathscr{C}_{\mathfrak{V}}(E)$.

(b) If E carries the weak topology induced by E', that is $\mathfrak{V}$ consists of the polars of all finite subsets of E', then the corresponding weighted topology on $\mathscr{C}_{\mathfrak{V}}(E)$ is generated by the semimetrics

$$d_\mu(A, B) = \left|\sup_{a \in A} \mu(a) - \sup_{b \in B} \mu(b)\right|$$

for $A, B \in \mathscr{C}_{\mathfrak{V}}(E)$ and $\mu \in E'$.

(c) Let E be an ordered normed space with unit ball $\mathbb{B}$, and for a convex subset A of E denote by $D(A)$ the decreasing closed and convex hull of A, i.e., the closure of the set $\{b \in E \mid b \leqslant a \text{ for some } a \in A\}$. Then $\mathfrak{V} = \{D(\mathbb{B})\}$ induces the topology of the Hausdorff metric with respect to $D(A)$ and $D(B)$ for sets $A, B \in \mathscr{C}_{\mathfrak{V}}(E)$.

We may use the family of neighborhoods $\mathfrak{V}$ to define an order relation on $\mathscr{C}_{\mathfrak{V}}(E)$: For elements $A, B \in \mathscr{C}_{\mathfrak{V}}(E)$ we denote

$$A \preccurlyeq_{\mathfrak{V}} B \qquad \text{if} \quad A \subset B + \varepsilon V \qquad \text{for all} \quad V \in \mathfrak{V} \qquad \text{and} \qquad \varepsilon > 0.$$

This relation on $\mathscr{C}_{\mathfrak{V}}(E)$ is in general finer than the set inclusion. Via the embedding as singleton sets it induces an order on E. By

$$E'_{\mathfrak{V}} = \bigcup \{ \rho V^{\circ} \mid V \in \mathfrak{V}, \rho > 0 \}$$

we denote the polar in E' of the negative cone in this order on E. Its non-zero elements are those functionals $\mu \in E'$ such that $(\mu, V) \in Y$ for some $V \in \mathfrak{V}$.

In order to apply Theorem 1.2 we consider linear (i.e., additive and positively homogeneous) operators T on subcones $\mathscr{D}_{\mathfrak{V}}(E)$ of $\mathscr{C}_{\mathfrak{V}}(E)$ satisfying:

(C′) *For every $V \in \mathfrak{V}$ there are $V' \in \mathfrak{V}$ and $\rho > 0$ such that for all $A, B \in \mathscr{D}_{\mathfrak{V}}(E)$*

$$T(A) \subset T(B) + V \qquad \textit{whenever} \quad A \subset B + \rho V'.$$

This requirement corresponds to condition (C) for the induced operators on the representation of $\mathscr{D}_{\mathfrak{V}}(E)$ as a subcone of $C_{\mathscr{V}}(Y)$. It combines continuity and monotony with respect to the order $\preccurlyeq_{\mathfrak{V}}$. Equicontinuity for a net $(T_\alpha)_{\alpha \in A}$ of such operators means that $V' \in \mathfrak{V}$ and $\rho > 0$ may be chosen simultaneously for all operators T_α.

Correspondingly, the *Korovkin closure* $\mathscr{K}(\mathscr{N})$ of a family $\mathscr{N} \subset \mathscr{D}_{\mathfrak{V}}(E)$ consists of all sets $A \in \mathscr{D}_{\mathfrak{V}}(E)$ such that $T_\alpha(A)$ converges to A, whenever $(T_\alpha)_{\alpha \in \mathscr{A}}$ is an equicontinuous net of linear operators on $\mathscr{D}_{\mathfrak{V}}(E)$ satisfying (C′), and $T_\alpha(N)$ converges to N for all $N \in \mathscr{N}$. Convergence is meant in the weighted topology of $\mathscr{D}_{\mathfrak{V}}(E)$. If $\mathscr{K}(\mathscr{N}) = \mathscr{D}_{\mathfrak{V}}(E)$, then $\mathscr{N}$ is a *Korovkin system* for $(\mathscr{D}_{\mathfrak{V}}(E), \mathfrak{V})$. Using the representation of $\mathscr{D}_{\mathfrak{V}}(E)$ as a subcone of $C_{\mathscr{V}}(Y)$, we shall proceed to identify Korovkin systems:

Only singleton sets $\{a\} \in \mathscr{C}_{\mathfrak{V}}(E)$ have an additive inverse $\{-a\} \in \mathscr{C}_{\mathfrak{V}}(E)$, and $f_{\{-a\}} = -f_{\{a\}}$ holds for the representing functions in $C_{\mathscr{V}}(Y)$. For a subset $\mathscr{N}$ of $\mathscr{C}_{\mathfrak{V}}(E)$ we define its *span* to be the subcone of $\mathscr{C}_{\mathfrak{V}}(E)$ generated by $\mathscr{N}$ and the negatives of all singleton sets in $\mathscr{N}$. By $\overline{\operatorname{span}}(\mathscr{N})$ we denote the closure of this span with respect to the weighted topology of $(\mathscr{C}_{\mathfrak{V}}(E), \mathfrak{V})$.

For $\mu \in E'_{\mathfrak{V}}$, the *half space*

$$H_\mu = \{ a \in E \mid \mu(a) \leqslant 0 \}$$

is closed in E and decreasing with respect to the order $\preccurlyeq_{\mathfrak{B}}$. The following criterion for Korovkin systems in $(\mathscr{D}_{\mathfrak{B}}(E), \mathfrak{B})$ corresponds to the criterion for Korovkin systems in $C_{\mathscr{W}}(X)$ in Proposition 1.3:

2.2. PROPOSITION. *Let $(\mathscr{D}_{\mathfrak{B}}(E), \mathfrak{B})$ be a weighted cone of sets, $\mathscr{N}$ a subset of $\mathscr{D}_{\mathfrak{B}}(E)$. Suppose that for every $\mu \in E'_{\mathfrak{B}}$ the union of all sets $D_{\mathfrak{B}}(N)$, for $N \in \overline{\text{span}}(\mathscr{N})$ such that $0 \in D_{\mathfrak{B}}(N) \subset H_\mu$, is dense in H_μ. Then $\mathscr{N}$ is a Korovkin system for $(\mathscr{D}_{\mathfrak{B}}(E), \mathfrak{B})$.*

Proof. We shall verify condition (c) from Theorem 1.2 for $C_{\mathscr{V}}(Y)$: Let $(\mu, V) \in Y$, that is, $0 \neq \mu \in E'_{\mathfrak{B}}$, and let $\phi \in M_B^+(Y)$ and $v \in \mathscr{V}$ such that $\int_Y v f_N \, d\phi = f_N(\mu, V)$ holds for all $N \in \mathscr{N}$, hence even for all $N \in \overline{\text{span}}(\mathscr{N})$. For any $(v, U) \in Y$ there is either an element a in the interior of H_μ such that $v(a) > 0$, or $H_\mu \subset H_v$, hence v is a positive multiple of μ. In the first case, by our assumption we may find $a' \in H_\mu$ such that $v(a') > 0$ as well, and $a' \in N$ for some $N \in \overline{\text{span}}(\mathscr{N})$ such that $0 \in D_{\mathfrak{B}}(N) \subset H_\mu$. We infer that the corresponding function f_N is non-negative on Y, $f_N(\mu, V) = 0$ and $f_N(v, U) \geqslant y(a') > 0$. As $\int_Y v f_N \, d\phi = 0$, this shows that any such point $(v, U) \in Y$ may not be contained in the support of the measure $v\phi$. This measure is therefore supported by the set

$$\{(v, U) \in Y \mid v = \lambda\mu \text{ for some } \lambda > 0\}.$$

But on this set any two functions f_A and f_B representing sets A and B in $\mathscr{D}_{\mathfrak{B}}(E)$ are proportional. The measure $v\phi$ therefore coincides on the representation of $\mathscr{D}_{\mathfrak{B}}(E)$ with a certain positive multiple $\rho\varepsilon_{(\mu, V)}$ of the point evaluation in (μ, V). Now we use our assumption for $v = 0 \in E'_{\mathfrak{B}}$: There is $a \in H_0 = E$ such that $\mu(a) > 0$ and $a \in N$ for some $N \in \overline{\text{span}}(\mathscr{N})$, hence $f_N(\mu, V) > 0$. But this yields $\int_Y v f_N \, d\phi = \rho\varepsilon_{(\mu, V)}(f_N) = f_N(\mu, V)$, demonstrating that $\rho = 1$, indeed. ∎

2.3. EXAMPLES. (a) If $\overline{\text{span}}(\mathscr{N})$ contains the segments $\{\lambda a \mid 0 \leqslant \lambda \leqslant 1\}$ for all $a \in E$, then $\mathscr{N}$ fulfills the criterion of Proposition 2.2, hence is a Korovkin system for $(\mathscr{C}_{\mathfrak{B}}(E), \mathfrak{B})$ for any choice of the family of weights $\mathfrak{B}$.

(b) For $E = \mathbb{R}^n$ with the unit vectors e_i and the Euclidean unit ball $\mathbb{B}$, the family

$$\mathscr{N} = \{\mathbb{B}, \{e_i\} \mid i = 1, ..., n\}$$

forms a Korovkin system for $(\mathscr{C}_{\mathfrak{B}}(\mathbb{R}^n), \mathfrak{B})$, where $\mathfrak{B} = \{\mathbb{B}\}$. This is a special case of the following:

(c) Let E be a separable real Hilbert space with the inner product $\langle\ ,\ \rangle$ and unit ball $\mathbb{B}$. Let $T\colon E \to E$ be a compact linear operator whose range is dense in E. The closure B of $T(\mathbb{B})$ is compact and convex. Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis for E. We claim that the family

$$\mathcal{N}=\{B, \{e_i\} \mid i\in\mathbb{N}\}$$

forms a Korovkin system for $(\mathscr{C}_{\mathfrak{B}}(E), \mathfrak{B})$, where $\mathfrak{B}=\{\mathbb{B}\}$. We shall verify the criterion of Proposition 2.2.

For $0\neq y\in E$, let $H_y=\{a\in E \mid \langle a, y\rangle \leqslant 0\}$ be a closed half space in E. The adjoint operator T^* of T is one-to-one, as the range of T is dense in E, hence $T^*(y)\neq 0$, and we set

$$b_0=\|T^*(y)\|^{-1}\, T(T^*(y))\in B.$$

As for all $b\in\mathbb{B}$

$$\langle T(b), y\rangle=\langle b, T^*(y)\rangle\leqslant\|T^*(y)\|=\langle b_0, y\rangle,$$

we infer that

$$\langle b_0, y\rangle=\max\{\langle b, y\rangle \mid b\in B\},$$

hence $0\in N=B-b_0\subset H_y$. Clearly $N\in\overline{\operatorname{span}}(\mathcal{N})$. Now a simple Hahn–Banach argument shows that the union of all positive multiples of N is indeed dense in H_y: Let $x\in E$ such that $\langle n, x\rangle\leqslant 0$ for all $n\in N$, i.e. $\langle b, x\rangle\leqslant\langle b_0, x\rangle$ for all $b\in B$. The choice of

$$b=\|T^*(x)\|^{-1}\, T(T^*(x))\in B,$$

for the above, together with the Cauchy–Schwarz inequality yields

$$\|T^*(x)\|\leqslant\|T^*(x)\|^{-1}\langle T^*(y), T^*(x)\leqslant\|T^*(x)\|.$$

But equality shows that the vectors $T^*(y)$ and $T^*(x)$ are linearly dependent, and as T^* is one-to-one, we infer that x is a multiple of y. Again using the density of the range of T, we conclude that this multiple is non-negative, and $\langle c, x\rangle\leqslant 0$ holds for all $c\in H_y$. Therefore H_y is contained in the closed subcone of E generated by N.

For $H_0=E$ we choose $N=B\in\mathcal{N}$ and observe that by our assumption on T the union of all positive multiples of B is dense in E.

(d) Let E be a real reflexive Banach space with unit ball $\mathbb{B}$. The norm on E is *smooth* if every point in the boundary of $\mathbb{B}$ is supported by a unique closed hyperplane (cf. [5, 20F]). In this case, let $\mathfrak{B}$ be a neighborhood base for the weak topology on E induced by E'. Then $\mathbb{B} \in \mathscr{C}_{\mathfrak{B}}(E)$. Let S be a total subset (i.e. the span of S is dense) of E. Then the family

$$\mathscr{N} = \{\mathbb{B}, \{e\} \mid e \in \mathscr{S}\}$$

forms a Korovkin system for $(\mathscr{C}_{\mathfrak{B}}(E), \mathfrak{B})$. We shall verify the criterion of Proposition 2.2.

For $0 \neq \mu \in E'$ let $H_\mu = \{a \in E \mid \mu(a) \leqslant 0\}$ be a closed half space in E. There is b_0 in the boundary of $\mathbb{B}$ such that

$$\mu(b_0) = \max\{\mu(b) \mid b \in B\},$$

hence $0 \in N = B - b_0 \subset H_\mu$ and $N \in \overline{\operatorname{span}}(\mathscr{N})$. As there is no other closed hyperplane H in E with this property, the union of all positive multiples of N is seen to be dense in H_μ. For $H_0 = E$ we choose all multiples of $N = \mathbb{B} \in \mathscr{N}$.

Examples of smoothly normed reflexive Banach spaces are integration spaces $L^p(X, \mu)$ for $1 < p < +\infty$.

(e) For $E = \mathbb{R}$ and $\mathfrak{B} = \{(-\infty, 1]\}$, all bounded above intervals in $\mathbb{R}$ are precompact relative to $\mathfrak{B}$, hence contained in $\mathscr{C}_{\mathfrak{B}}(\mathbb{R})$. A Korovkin system for $(\mathscr{C}_{\mathfrak{B}}(\mathbb{R}), \mathfrak{B})$ is given by $\mathscr{N} = \{(-\infty, 1]\}$.

2.4. *Remark.* Readers familiar with the theory of locally convex cones as developed in [7] will realize that a similar representation may be used in this more general situation: A locally convex cone (P, V) may be embedded in a full cone $(\tilde{P}, V)$ that contains all neighborhoods as elements. The dual cone of $\tilde{P}$ is locally compact in its weak* topology and serves as the domain for a suitable weighted space of real valued functions $C_{\mathscr{V}}(Y)$. The weight functions are the characteristic functions of the polars of neighborhoods. If all its elements are bounded, then P is canonically represented as a subcone of $C_{\mathscr{V}}(Y)$.

3. WEIGHTED CONES OF SET-VALUED FUNCTIONS

As before, let X be a locally compact Hausdorff space and $\mathscr{W}$ a family of weights on X. Let E be a real locally convex topological vector space and $\mathfrak{B}$ a family of closed convex neighborhoods of the origin in E, directed downward for set inclusion. Let $\mathscr{C}_{\mathfrak{B}}(E)$ be the cone of all non-empty convex subsets of E that are precompact relative to $\mathfrak{B}$, $\mathscr{D}_{\mathfrak{B}}(E)$ a subcone of $\mathscr{C}_{\mathfrak{B}}(E)$.

A function $F\colon X \to \mathscr{D}_{\mathfrak{V}}(E)$ is *continuous at* $x_0 \in X$ if for every $V \in \mathfrak{V}$ and $\varepsilon > 0$ there is a neighborhood U of x_0 such that $d_V(F(x), F(x_0)) \leqslant \varepsilon$, that is,

$$F(x) \subset F(x_0) + \varepsilon V \qquad \text{and} \qquad F(x_0) \subset F(x) + \varepsilon V$$

for all $x \in U$. Let $C(X, \mathscr{D}_{\mathfrak{V}}(E))$ denote the cone of all such functions that are continuous on X. Given $V \in \mathfrak{V}$, we say that a function $F\colon X \to \mathscr{D}_{\mathfrak{V}}(E)$ *vanishes at infinity relative to* V if for every $\varepsilon > 0$ the set $\{x \in X \mid F(x) \not\subset \varepsilon V\}$ is relatively compact in X. Accordingly, we denote by $C_{\mathscr{W} \otimes \mathfrak{V}}(X, \mathscr{D}_{\mathfrak{V}}(E))$ the cone of all functions $F \in C(X, \mathscr{D}_{\mathfrak{V}}(E))$ such that wF vanishes at infinity relative to V for all $V \in \mathfrak{V}$ and $w \in \mathscr{W}$. The weighted topology on $C_{\mathscr{W} \otimes \mathfrak{V}}(X, \mathscr{D}_{\mathfrak{V}}(E))$ is induced by the semimetrics

$$d_{w, V}(F, G) = \sup_{x \in X} \{w(x)\, d_V(F(x), G(x))\}$$

for $F, G \in C_{\mathscr{W} \otimes \mathfrak{V}}(X, \mathscr{D}_{\mathfrak{V}}(E))$, $w \in \mathscr{W}$ and $V \in \mathfrak{V}$. We call $C_{\mathscr{W} \otimes \mathfrak{V}}(X, \mathscr{D}_{\mathfrak{V}}(E))$ a *weighted cone of set-valued functions.* Simple compactness arguments show that for any real-valued function $f \in C_{\mathscr{W}}(X)$ and a set $A \in \mathscr{D}_{\mathfrak{V}}(E)$ the set-valued function

$$x \mapsto f(x)\, A$$

belongs to $C_{\mathscr{W} \otimes \mathfrak{V}}(X, \mathscr{D}_{\mathfrak{V}}(E))$. For any function $F \in C_{\mathscr{W} \otimes \mathfrak{V}}(X, \mathscr{D}_{\mathfrak{V}}(E))$, all $w \in \mathscr{W}$, $V \in \mathfrak{V}$, and $\varepsilon > 0$ one may find a set $C \in \mathscr{C}_{\mathfrak{V}}(E)$ such that

$$w(x)\, F(x) \subset C + \varepsilon V \qquad \text{for all} \quad x \in X.$$

For Korovkin type approximation problems we consider linear (i.e., additive and positively homogeneous) operators T on $C_{\mathscr{W} \otimes \mathfrak{V}}(X, \mathscr{D}_{\mathfrak{V}}(E))$ satisfying

(C″) *For all $w \in \mathscr{W}$ and $V \in \mathfrak{V}$ there are $w' \in \mathscr{W}$, $V' \in \mathfrak{V}$ and $\rho > 0$ such that for all $F, G \in C_{\mathscr{W} \otimes \mathfrak{V}}(X, \mathscr{D}_{\mathfrak{V}}(E))$*

$$w(x)\, T(F)(x) \subset w(x)\, T(G)(x) + V \qquad \textit{for all} \quad x \in X,$$

whenever

$$w'(x)\, F(x) \subset w'(x)\, G(x) + \rho V' \qquad \textit{for all} \quad x \in X,$$

Evidently, this condition combines continuity and monotony. Equicontinuity for a net $(T_\alpha)_{\alpha \in A}$ of such operators means that w', V' and ρ may be chosen simultaneously for all operators T_α.

For a subset $\mathscr{L}$ of $C_{\mathscr{W}\otimes\mathfrak{B}}(X, \mathscr{D}_{\mathfrak{B}}(E))$, the *Korovkin closure* $\mathscr{K}(\mathscr{L})$ of $\mathscr{L}$ consists of all functions $F\in C_{\mathscr{W}\otimes\mathfrak{B}}(X, \mathscr{D}_{\mathfrak{B}}(E))$ such that $T_\alpha(F)$ converges to F, whenever $(T_\alpha)_{\alpha\in\mathscr{A}}$ is an equicontinuous net of linear operators on $C_{\mathscr{W}\otimes\mathfrak{B}}(X, \mathscr{D}_{\mathfrak{B}}(E))$ satisfying (C″), and $T_\alpha(L)$ converges to L for all $L\in\mathscr{L}$. Convergence is meant in the weighted topology of $C_{\mathscr{W}\otimes\mathfrak{B}}(X, \mathscr{D}_{\mathfrak{B}}(E))$.

We shall use the representation of $(\mathscr{D}_{\mathfrak{B}}(E), \mathfrak{B})$ as a subcone of $C_{\mathscr{V}}(Y)$ in order to represent $C_{\mathscr{W}\otimes\mathfrak{B}}(X, \mathscr{D}_{\mathfrak{B}}(E))$ as a subcone of a weighted space of real-valued functions on $X\times Y$. We begin with some general observations.

For real-valued functions f on X and g on Y we shall denote by $f\otimes g$ the function $(x, y)\mapsto g(x)\, f(y)$ on $X\times Y$. For sets $\mathscr{M}$ and $\mathscr{N}$ of real-valued functions on X and Y we set

$$\mathscr{M}\otimes\mathscr{N}=\{m\otimes n \mid m\in\mathscr{M}, n\in\mathscr{N}\}.$$

The function $f\otimes g$ is continuous on $X\times Y$ if both f and g are continuous on X and Y, upper resp. lower semicontinuous if both f and g are upper resp. lower semicontinuous and non-negative. Thus, if $\mathscr{W}$ and $\mathscr{V}$ are families of weights on X and Y respectively, then $\mathscr{W}\otimes\mathscr{V}$ is a family of weights on $X\times Y$. It is straightforward to check that

$$C_{\mathscr{W}}(X)\otimes C_{\mathscr{V}}(Y)\subset C_{\mathscr{W}\otimes\mathscr{V}}(X\times Y).$$

For the seminorms on $C_{\mathscr{W}\otimes\mathscr{V}}(X\times Y)$, note that

$$p_{w\otimes v}(f\otimes g)=p_w(f)\, p_v(g).$$

As before we denote by

$$X_0=\{x\in X \mid w(x)>0 \text{ for some } w\in\mathscr{W}\}$$

and

$$Y_0=\{y\in Y \mid v(y)>0 \text{ for some } v\in\mathscr{V}\}.$$

For compact spaces X and Y and the unit families $\mathscr{E}=\{1\}$ and $\mathscr{F}=\{1\}$ the following may be found in [6]:

3.1. THEOREM. *Let X and Y be locally compact Hausdorff spaces, $\mathscr{W}$ and $\mathscr{V}$ families of weights on X and Y, respectively. Let $\mathscr{M}$ be a Korovkin system, $\mathscr{E}$ a unit family for $C_{\mathscr{W}}(X)$. Let $\mathscr{N}$ be a subset, $\mathscr{F}$ a unit family for $C_{\mathscr{V}}(Y)$. Set*

$$\mathscr{L}=(\mathscr{M}\otimes\mathscr{F})\cup(\mathscr{E}\otimes\mathscr{N}).$$

If for $F \in C_{\mathscr{W} \otimes \mathscr{V}}(X \times Y)$ *the functions* $y \mapsto F(x, y)$ *are in* $\mathscr{K}(\mathscr{N})$ *for every* $x \in X_0$, *then* $F \in \mathscr{K}(\mathscr{L})$. *In particular, if* $\mathscr{N}$ *is a Korovkin system for* $C_{\mathscr{V}}(Y)$, *then* $\mathscr{L}$ *is a Korovkin system for* $C_{\mathscr{W} \otimes \mathscr{V}}(X \times Y)$.

Proof. We shall verify condition (c) in Theorem 1.2. Let $(x_0, y_0) \in X_0 \times Y_0$, and let $\phi \in M_B^+(X \times Y)$ and $w \otimes v \in \mathscr{W} \otimes \mathscr{V}$ such that

$$\int_{X \times Y} (w \otimes v)\, L\, d\phi = L(x_0, y_0) \qquad \text{for all} \quad L \in \mathscr{L}.$$

For any non-negative function $f \in \operatorname{span}(\mathscr{F})$ such that $f(y_0) = 1$, the mapping

$$g \mapsto g \otimes f : C_{\mathscr{W}}(X) \to C_{\mathscr{W} \otimes \mathscr{V}}(X \times Y)$$

is an embedding that preserves the seminorms of $C_{\mathscr{W}}(X)$, as for $w' \otimes v' \in \mathscr{W} \otimes \mathscr{V}$ we have $p_{w' \otimes v'}(g \otimes f) = p_{w'}(g)\, p_{v'}(f)$, and as $p_{v'}(f) > 0$ for some $v' \in V$. Thus

$$g \mapsto \int_{X \times Y} (w \otimes v)(g \otimes f)\, d\phi$$

defines a continuous positive linear functional on $C_{\mathscr{W}}(X)$ which coincides with the point evaluation in x_0 on the Korovkin system $\mathscr{M}$, hence on all of $C_{\mathscr{W}}(X)$ by Theorem 1.2(c). Now for any $x_0 \neq x \in X$ there is a non-negative function $g \in C_{\mathscr{W}}(X)$ such that $g(x) > g(x_0) = 0$, and for any $y \in Y_0$ there is a non-negative function $f \in \operatorname{span}(\mathscr{F})$ such that $f(y_0) = 1$ and $f(y) > 0$. The above shows that none of these points (x, y) may be in the support of the Borel measure $(w \otimes v)\, \phi$. As $v(y) = 0$ for the remaining points $y \in Y$, we realize that this measure is in fact supported by the set $\{x_0\} \times Y \subset X \times Y$. Next we choose a non-negative function $e \in \operatorname{span}(\mathscr{E})$ such that $e(x_0) = 1$. The mapping

$$g \mapsto e \otimes g : C_{\mathscr{V}}(Y) \mapsto C_{\mathscr{W} \otimes \mathscr{V}}(X \times Y)$$

is an embedding that preserves the seminorms of $C_{\mathscr{V}}(Y)$, and

$$g \mapsto \int_{X \times Y} (w \otimes v)(e \otimes g)\, d\phi$$

is a continuous positive linear functional on $C_{\mathscr{V}}(Y)$ that evaluates as $n(y_0)$ for all $n \in \mathscr{N}$. Thus, as the function f, mapping $y \mapsto F(x_0, y)$, is contained in $\mathscr{K}(\mathscr{N})$, and as the functions F and $e \otimes f$ coincide on the set $\{x_0\} \times Y$, we have

$$\int_{X \times Y} (w \otimes v)\, F\, d\phi = \int_{X \times Y} (w \otimes v)(e \otimes f)\, d\phi = f(y_0) = F(x_0, y_0),$$

indeed. ▮

For our purposes we choose the locally compact space Y and the family $\mathscr{V}$ weight functions corresponding to the neighborhood system $\mathfrak{V}$ as in Section 2. With every set-valued function $F \in C_{\mathscr{W} \otimes \mathfrak{V}}(X, \mathscr{D}_{\mathfrak{V}}(E))$ we associate the real-valued function f_F on $X \times Y$ such that for $x \in X$ and $y = (\mu, V) \in Y$

$$f_F(x, y) = f_{F(x)}(\mu, V) = \sup \{\mu(a) \mid a \in F(x)\}.$$

We first verify that f_F is continuous on $X \times Y$: Let $x_0 \in X$ and $y_0 = (\mu, V) \in Y$. Given $\varepsilon > 0$, from the continuity of the function $f_{F(x_0)}$ on Y we infer that there is a neighborhood $U_Y \subset Y_V$ of y_0 such that

$$|f_F(x_0, y) - f_F(x_0, y_0)| = |f_{F(x_0)}(y) - f_{F(x_0)}(y_0)| \leqslant \varepsilon/2$$

holds for all $y \in U_Y$. Likewise, using the neighborhood $V \in \mathfrak{V}$ from above, there is a neighborhood U_X of x_0 such that

$$d_V(F(x), F(x_0)) \leqslant \varepsilon/2$$

for all $x \in U_X$. Then for all $(x, y) \in U_X \times U_Y$

$$|f_F(x, y) - f_F(x_0, y)| = |f_{F(x)}(\mu, V) - f_{F(x_0)}(\mu, V)| \leqslant \varepsilon/2.$$

Thus

$$\begin{aligned} |f_F(x, y) - f_F(x_0, y_0)| &\leqslant |f_F(x, y) - f_F(x_0, y)| + |f_F(x_0, y) - f_F(x_0, y_0)| \\ &\leqslant \varepsilon. \end{aligned}$$

Next we shall show that f_F belongs to $C_{\mathscr{W} \otimes \mathscr{V}}(X \times Y)$ whenever F belongs to $C_{\mathscr{W} \otimes \mathfrak{V}}(X, \mathscr{D}_{\mathfrak{V}}(E))$. Indeed, for $w \in \mathscr{W}$, the weight function $v \in \mathscr{V}$ corresponding to $V \in \mathfrak{V}$ and $\varepsilon > 0$,

$$(w \otimes v)\, f_F(x, y) = v(y)\, w(x)\, f_{F(x)}(y) \geqslant \varepsilon$$

for $(x, y) \in X \times Y$ implies that $y = (\mu, V) \in Y_V$, hence $\mu \in V^\circ$ and $w(x)\, F(x) \not\subset (\varepsilon/2)\, V$. Therefore x belongs to the relatively compact subset

$$X_c = \{x \in X \mid w(x)\, F(x) \not\subset (\varepsilon/2)\, V\}$$

of X. As mentioned before, one may find $C \in \mathscr{C}_{\mathfrak{V}}(E)$ such that $w(x)\, F(x) \subset C + (\varepsilon/2)\, V$ for all $x \in X$. Accordingly,

$$\varepsilon \leqslant v(y)\, w(x)\, f_{F(x)}(y) \leqslant f_C(y) + \varepsilon/2$$

shows that y belongs to

$$Y_c = \{(v, V) \in Y_V \mid f_C(y) \geqslant \varepsilon/2\}.$$

This set is compact, as f_C is continuous on Y, demonstrating that (x, y) belongs to the relatively compact product $X_c \times Y_c$, indeed. The embedding

$$F \mapsto f_F : C_{\mathscr{W} \otimes \mathfrak{V}}(X, \mathscr{D}_{\mathfrak{V}}(E)) \to C_{\mathscr{W} \otimes \mathscr{V}}(X \times Y)$$

is additive, positively homogeneous, and monotone if we endow $C_{\mathscr{W} \otimes \mathfrak{V}}(X, \mathscr{D}_{\mathfrak{V}}(E))$ with the order of pointwise set inclusion. Furthermore, for $F, G \in C_{\mathscr{W} \otimes \mathfrak{V}}(X, \mathscr{D}_{\mathfrak{V}}(E))$ and $\rho \geqslant 0$ we have

$$(w \otimes v)\, f_F(x, y) \leqslant (w \otimes v)\, f_G(x, y) + \rho$$

for all $(x, y) \in X \times Y$ if and only if

$$w(x)\, F(x) \subset w(x)\, G(x) + \rho' V$$

for all $x \in X$ and $\rho' > \rho$. Thus, linear operators on $C_{\mathscr{W} \otimes \mathfrak{V}}(X, \mathscr{D}_{\mathfrak{V}}(E))$ satisfying (C″) correspond to linear operators on a subcone of $C_{\mathscr{W} \otimes \mathscr{V}}(X \times Y)$ satisfying (C).

Propositions 1.3 and 2.2 provide criteria to identify Korovkin systems in $C_{\mathscr{W}}(X)$ and $\mathscr{D}_{\mathfrak{V}}(E)$, the latter being identified with a subcone of $C_{\mathscr{V}}(Y)$. Unit families in $C_{\mathscr{W}}(X)$ are obvious and may consist of a single strictly positive function. A unit family $\mathscr{F}$ in $\mathscr{D}_{\mathfrak{V}}(E)$ consists of sets $F \in \mathscr{D}_{\mathfrak{V}}(E)$ such that $0 \in D_{\mathfrak{V}}(F)$. This guarantees that the function f_F is non-negative on Y. We require that for every $y = (\mu, V) \in Y$ there is $F \in \mathscr{F}$ such that $f_F(y) > 0$. The latter holds in particular if the union of all non-negative multiples of the sets in $\mathscr{F}$ is dense in E.

Summarizing, we may now formulate our result. For compact spaces X, $E = \mathbb{R}^n$, the unit families $\mathscr{E} = \{1\}$ and $\mathscr{F} = \{\mathbb{B}\}$, and $\mathscr{N}$ consisting of all non-empty compact convex subsets of $\mathbb{R}^n$, it may be found in [6]. For paracompact spaces X and Frechét spaces E, related characterizations of Korovkin systems for set-valued functions had been established in [3, 4].

3.2. Theorem. *Let $C_{\mathscr{W} \otimes \mathfrak{V}}(X, \mathscr{D}_{\mathfrak{V}}(E))$ be a weighted cone of set-valued functions. Let $\mathscr{M}$ be a Korovkin system of non-negative functions, $\mathscr{E}$ a unit* *family for* $C_{\mathscr{W}}(X)$, *and let* $\mathscr{N}$ *be a Korovkin system,* $\mathscr{F}$ *a unit family for* $(\mathscr{D}_{\mathfrak{B}}(E), \mathfrak{B})$. *Then the functions*

$$x \mapsto m(x)\, F \quad \text{for} \quad m \in \mathscr{M} \quad \text{and} \quad F \in \mathscr{F}$$

$$x \mapsto e(x)\, N \quad \text{for} \quad e \in \mathscr{E} \quad \text{and} \quad N \in \mathscr{N}$$

form a Korovkin system for $C_{\mathscr{W} \otimes \mathfrak{B}}(X, \mathscr{D}_{\mathfrak{B}}(E))$.

This follows directly from Theorem 3.1, if we keep in mind that in the preceding representation, for $g \in C_{\mathscr{W}}(X)$ and $A \in \mathscr{D}_{\mathfrak{B}}(E)$ the set-valued function $x \mapsto g(x)\, A$ in $C_{\mathscr{W} \otimes \mathfrak{B}}(X, \mathscr{D}_{\mathfrak{B}}(E))$ corresponds to the real-valued function $g \otimes f_A$ in $C_{\mathscr{W} \otimes \mathscr{V}}(X \times Y)$.

3.3. Examples. (a) For $X = [0, 1]$ and $\mathscr{W} = \{1\}$ we may choose the classical Korovkin system $\mathscr{M} = \{1, x, x^2\}$ and the unit family $\mathscr{E} = \{1\}$. For $E = \mathbb{R}^n$ and $\mathfrak{B} = \{\mathbb{B}\}$ we may choose $\mathscr{N} = \{\mathbb{B}, \{e_i\} \mid i = 1, ..., n\}$ (cf. Example 2.3(b)) and $\mathscr{F} = \{\mathbb{B}\}$. In this way Theorem 3.2 yields Vitale's result [11].

(b) Let $X = [0, +\infty)$, and let $\mathscr{W}$ consist of the functions $w_\alpha(x) = e^{-\alpha x}$ for all $\alpha > 0$. Following Proposition 1.3, the subset

$$\mathscr{M} = \{m_k \mid m_k(x) = x^k \text{ for } k = 0, 1, 2\}$$

is a Korovkin system for $C_{\mathscr{W}}(X)$. We choose $\mathscr{E} = \{1\}$. For a normed space E and $\mathfrak{B} = \{\mathbb{B}\}$, the family $\mathscr{N} = \mathscr{F}$ of all segments $A = \{\lambda a \mid 0 \leqslant \lambda \leqslant 1\}$, for $a \in E$, forms a Korovkin system as well as a unit family for $\mathscr{C}_{\mathfrak{B}}(E)$ (Example 2.3(a)). With these insertions Theorem 3.3 describes a suitable Korovkin system for $C_{\mathscr{W} \otimes \mathfrak{B}}(X, \mathscr{C}_{\mathfrak{B}}(E))$.

Let us illustrate this example with an approximation process modeled by a modified version of the classical Bernstein operators. For a function F in $C_{\mathscr{W} \otimes \mathfrak{B}}(X, \mathscr{C}_{\mathfrak{B}}(E))$ and $n \in \mathbb{N}$ define

$$T_n(F)(x) = \begin{cases} \displaystyle\sum_{k=0}^{n^2} \binom{n^2}{k} \left(\frac{x}{n}\right)^k \left(1 - \frac{x}{n}\right)^{n^2 - k} F\left(\frac{k}{n}\right), & \text{for} \quad x < n \\ F(n) & \text{for} \quad x \geqslant n. \end{cases}$$

These operators T_n are linear on $C_{\mathscr{W} \otimes \mathfrak{B}}(X, \mathscr{C}_{\mathfrak{B}}(E))$. With some straightforward computations one may check the following: for every $A \in \mathscr{C}_{\mathfrak{B}}(E)$

$$T_n(m_0 \otimes A)(x) = A \qquad \text{for all} \quad x \in [0, +\infty),$$

$$T_n(m_1 \otimes A)(x) = xA \qquad \text{for all} \quad x < n,$$

$$T_n(m_2 \otimes A)(x) = \left(\frac{n^2 - 1}{n^2}\, x^2 + \frac{1}{n}\, x\right) A \qquad \text{for all} \quad x < n.$$

This shows in particular, as A is bounded, that $T_n(m_k \otimes A)$ converges to $m_k \otimes A$ for $k=0, 1, 2$ in the topology of $C_{\mathscr{W} \otimes \mathfrak{V}}(X, \mathscr{C}_{\mathfrak{V}}(E))$. Furthermore, one may check that the sequence $(T_n)_{n \in \mathbb{N}}$ satisfies (C″), as for all $F, G \in C_{\mathscr{W} \otimes \mathfrak{V}}(X, \mathscr{C}_{\mathfrak{V}}(E))$ and $\alpha > 0$ and all $n \in \mathbb{N}$

$$F(x) \subset G(x) + e^{\alpha x}\mathbb{B} \qquad \text{for all} \quad x \in X$$

implies

$$T_n(F)(x) \subset T_n(G)(x) + e^{(e\alpha)\,x}\mathbb{B} \qquad \text{for all} \quad x \in X.$$

We conclude that $T_n(F)$ converges to F for all $F \in C_{\mathscr{W} \otimes \mathfrak{V}}(X, \mathscr{C}_{\mathfrak{V}}(E))$.

(c) For $X = \mathbb{N}$ and $\mathscr{W} = \{1\}$, $C_{\mathscr{W}}(X)$ is the space c_0 of all sequences $(x_i)_{i \in \mathbb{N}}$ in $\mathbb{R}$ converging to 0, endowed with the l^∞-norm. The family

$$\mathscr{M} = \{(1/i^k)_{i \in \mathbb{N}} \mid \text{for } k = 1, 2, 3\}$$

fulfills the criterion of Proposition 1.3 and forms a Korovkin system for c_0. We choose $\mathscr{E} = \{(1/i)_{i \in \mathbb{N}}\}$. For a separable Hilbert space E and an orthonormal basis $\{e_i\}_{i \in \mathbb{N}}$, the linear operator T such that $T(e_i) = e_i/i$, is compact and its range is dense in E. Following Example 2.3(c), the set

$$\mathscr{N} = \{B, \{e_i\} \mid i \in \mathbb{N}\},$$

where B is the closure of $T(\mathbb{B})$, forms a Korovkin system for $(\mathscr{C}_{\mathfrak{V}}(E), \mathfrak{V})$, where $\mathfrak{V} = \{\mathbb{B}\}$. We choose $\mathscr{F} = \{B\}$. The cone

$$C_{\mathscr{W} \otimes \mathfrak{V}}(X, \mathscr{C}_{\mathfrak{V}}(E)) = c_0(\mathscr{C}_{\mathfrak{V}}(E))$$

consists of all sequences $(A_i)_{i \in \mathbb{N}}$ of non-empty compact convex subsets of E converging to $\{0\}$ with respect to the Hausdorff metric, endowed with the topology of uniform convergence. Using the above insertions for $\mathscr{M}$, $\mathscr{N}$, $\mathscr{E}$, and $\mathscr{F}$, Theorem 3.3 describes a suitable Korovkin system for $c_0(\mathscr{C}_{\mathfrak{V}}(E))$.

For a concrete approximation process, let P_n denote the orthogonal projection of E onto the span of $\{e_1, ..., e_n\}$, and for $A \in \mathscr{C}_{\mathfrak{V}}(E)$ set

$$P_n(A) = \{P_n(a) \mid a \in A\} \in \mathscr{C}_{\mathfrak{V}}(E).$$

We abbreviate B_n for $P_n(B)$ and observe that $B_n \subset B \subset B_n + (1/n)\,\mathbb{B}$. Now we define linear operators T_n on $c_0(\mathscr{C}_{\mathfrak{V}}(E))$ as follows: For $(A_i)_{i \in \mathbb{N}} \in c_0(\mathscr{C}_{\mathfrak{V}}(E))$ set

$$T_n((A_i)_{i \in \mathbb{N}}) = (P_n(A_i + A_{i+n}))_{i \in \mathbb{N}}.$$

These operators satisfy (C″), as for all $(A_i)_{i\in\mathbb{N}}, (C_i)_{i\in\mathbb{N}} \in c_0(\mathscr{C}_{\mathfrak{B}}(E))$

$$A_i \subset C_i + \mathbb{B} \qquad \text{for all} \quad i \in \mathbb{N}$$

implies

$$P_n(A_i + A_{i+n}) \subset P_n(C_i + C_{i+n}) + 2\mathbb{B} \qquad \text{for all} \quad i \in \mathbb{N}.$$

We claim that $T_n((A_i))$ converges to (A_i) for all $(A_i) \in c_0(\mathscr{C}_{\mathfrak{B}}(E))$. For the sequences in our Korovkin system we have for $k = 1, 2, 3$

$$T_n\left(\left(\frac{1}{i^k} B\right)_{i\in\mathbb{N}}\right) = \left(\left(\frac{1}{i^k} + \frac{1}{(i+n)^k}\right) B_n\right)_{i\in\mathbb{N}},$$

thus for every $i \in \mathbb{N}$

$$\left(\frac{1}{i^k} + \frac{1}{(i+n)^k}\right) B_n \subset \frac{1}{i^k} B + \frac{1}{n} \mathbb{B}$$

and

$$\frac{1}{i^k} B \subset \left(\frac{1}{i^k} + \frac{1}{(i+n)^k}\right) B_n + \frac{1}{n} \mathbb{B}.$$

This shows convergence of the operators T_n toward the identity for the sequences in $\mathscr{M} \otimes \mathscr{F}$. This convergence is obvious for the sequences $(\{e_k/i\})_{i\in\mathbb{N}}$ in $\mathscr{E} \otimes \mathscr{N}$, hence our claim follows.

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